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# On Survey of the Some Wave Solutions of the Non-Linear Schrödinger Equation (NLSE) in Infinite Water Depth

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**Highlights**

- This paper focuses on two different analytic schemes.
- We have describe gravity waves in infinite deep water, in the sense of conformable derivative.
- We have drawn the 2D-3D and contour surfaces under the appropriate values of constants.

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**Abstract**

In this work, we use two different analytic schemes which are the Sine-Gordon expansion technique and the modified exp $\left(-\Omega(\zeta)\right)$-expansion function technique to construct novel exact solutions of the non-linear Schrödinger equation, describing gravity waves in infinite deep water, in the sense of conformable derivative. After getting various travelling wave solutions, we plot 3D, 2D and contour surfaces to present behaviours obtained exact solutions.

## 1. INTRODUCTION

The NLSE is an intermediary wave function that allows us to conclude in the analysis of a quantum system. Quantum mechanics calculate the probability of a particle at a certain location or the probability of having a certain momentum. It realizes the possibility with the help of a wave function. The purpose of this function is not to find the location but to calculate the probability of the position and the NLSE is one of them. It is very practical to study the NLSE equation in spherical coordinates if the potential of a physical system has a spherically symmetrical distribution. The NLSE is an equation that shows the change in space and time.

The NLSE is a usually used equation in physical science. An example of how useful the NLSE is how optical pulses are propagated in fibres. The NLSE is used to model telecommunications, hydrodynamics, non-linear acoustics, non-linear dispersive waves, plasmas, optics, water waves, and the dynamics of particles [1].

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Someone the important improvement in mathematics in the last years has been the solution of certain types of the NLSE. The numeric simulation and analytic types of the NLSE plays a significant role in the design optimization of optic transmission systems. Many authors have researched analytic and numeric solutions of the NLSE and other governing equations using different methods. Some of methods are the inverse scattering transform technique [2], the extended sinh Gordon equation expansion technique [3], the improved Bernoulli sub-equation function technique [4, 5], the homotopy analysis technique [6], the variational principle [7], the generalized Kudryashov technique [8], the extended tanh function technique [9], the split-step technique [10], the finite difference technique [11], the variable separated ODE technique [12], the Fourier pseudospectral technique [13].

When many events in nature and interdisciplinary sciences are modelled mathematically, they are defined by non-linear partial differential equations. Therefore, many scientists have emphasized on the soliton solutions of non-linear, especially partial differential formation equations of late years. Today, different types of solitons have been observed both experimentally and theoretically in science. The NLSE accept solutions that are usually known as solitons or self-reinforcing waves that preserve form and velocity during spread. Varied types of solitons form when the non-linear term of the NLSE cancels with the distribution terms. Soliton solutions were obtained both analytically and later work on the soliton was accelerated. As the soliton, the solitary wave noticed in a water channel first today. It is used in many fields of physics like fluid mechanics, fundamental particle physics, biophysics. The NLSE is seen in non-linear optics, hydromagnetic and plasma waves and such [14-20] .

In this study, we will examine some wave solutions of the NLSE handling the Sine Gordon Expansion technique and the modified $\exp(-\Omega(\zeta))$-expansion function technique.

The NLSE describing gravity waves in deep water. It is given as in the literature [21-25],

$$i\left(\frac{\partial U}{\partial t}+c_g\frac{\partial U}{\partial x}\right)-\frac{\omega_0}{8k_0^2}\frac{\partial^2 U}{\partial x^2}-\frac{\omega_0 k_0^2}{2}|U|^2U=0, \tag{1}$$

where $t$ and $x$ are the time and longitudinal coordinates when $k_0$ and $\omega_0=\omega(k_0)$ signify the number of wave and wave frequency, respectively. Here, $\omega_0=\sqrt{gk_0}$ and where $g$ is gravity acceleration and $c_g=\frac{d\omega}{dk}=\frac{\omega_0}{2k_0}$ is group velocity.

In this work, we focus on finding solitary wave solutions of Equation (1) in conformable sense.

## 2. PRELIMINARIES

**Definition 2.1.** Let $h:[0,\infty)\to\mathbb{R}$ be a given function, the conformable derivative of $h$ of order $\alpha$ is defined as,

$$L_\alpha(h)(t)=\lim_{\varepsilon\to 0}\frac{h\left(t+\varepsilon t^{1-\alpha}\right)-h(t)}{\varepsilon},$$

for all $t>0$, $\alpha\in(0,1]$ [26].

**Theorem 2.2.** Let $L_\alpha$ be the derivative operator with order $\alpha$ and $\alpha \in (0,1]$ and $h,k$ be $\alpha$-differentiable at a point $t > 0$. Then [26,27], we can write the following properties

i. $L_\alpha(ah+bk) = aL_\alpha(h) + bL_\alpha(k), \forall a,b \in \mathbb{R}.$

ii. $L_\alpha(t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R}.$

iii. $L_\alpha(hk) = hL_\alpha(g) + kL_\alpha(f).$

iv. $L_\alpha\left(\frac{h}{k}\right) = \frac{kL_\alpha(h) - hL_\alpha(k)}{k^2}.$

v. $L_\alpha(\lambda) = 0,$ for all constant functions $h(t) = \lambda.$

vi. If $h$ is differentiable then $L_\alpha(h)(t) = t^{1-\alpha}\frac{dh}{dt}(t).$

**Proposition 2.3.** Let $L_\alpha$ be the derivative operator with order $\alpha$ and $\alpha \in (0,1]$. Then

1. $L_\alpha(1) = 0.$
2. $L_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}, c \in \mathbb{R}.$
3. $L_\alpha(\sin bx) = bx^{1-\alpha}\cos bx, b \in \mathbb{R}.$
4. $L_\alpha(\cos bx) = -bx^{1-\alpha}\sin bx, b \in \mathbb{R}.$
5. $L_\alpha\left(\frac{t^\alpha}{\alpha}\right) = 1.$

## 3. MATERIAL METHOD

### 3.1. Fundamental Properties of SGEM

In this part, we define the SGEM. We need two important equations prior to giving the common properties of Sine Gordon Equations [28, 29].

Primarily, let's presume that the Sine Gordon equation is given as following [30,31,32];

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{2}$$

where $u = u(x,t)$, $m$ is a real fixed. Implementing the wave transform $u = u(x,t) = U(\xi),\ \xi = \mu(x - ct)$ to Equation (2),

$$\begin{aligned} u_x &= \frac{dU}{d\xi}.\frac{d\xi}{dx} = \mu.U', u_{xx} = \frac{d(u_x)}{d\xi}.\frac{d\xi}{dx} = \mu^2 U'', \\ u_t &= \frac{dU}{d\xi}.\frac{d\xi}{dt} = -\mu.c.U', u_{tt} = \frac{d(u_t)}{d\xi}.\frac{d\xi}{dt} = c^2\mu^2 U'', \end{aligned} \tag{3}$$

Equation (3) is acquired. After putting Equation (3) into Equation (2) and when necessary arrangements are made, we acquire the following non-linear ordinary differential equation;

$$U'' = \frac{m^2}{\mu^2\left(1-c^2\right)}\sin\left(U\right), \tag{4}$$

where $U = U(\xi), \xi$ is the amplitude of the travelling wave and $c$ is the speed of the travelling wave. Equation (4) can be written as follows;

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{m^2}{\mu^2\left(1-c^2\right)}\sin^2\left(\frac{U}{2}\right)+K, \tag{5}$$

where $K$ is the constant of integration. Substituting $K = 0, w\left(\xi\right) = \frac{U}{2}$ and $a^2 = \frac{m^2}{\mu^2\left(1-c^2\right)}$ in Equation (5), gives;

$$w' = a\sin\left(w\right), \tag{6}$$

setting $a = 1$ in Equation (6), gives;

$$w' = \sin\left(w\right). \tag{7}$$

If Equation (7) is solved by the method of separation of variables, we get the following two important properties.;

$$\sin(w) = \sin\left(w\left(\xi\right)\right) = \left.\frac{2pe^{\xi}}{p^2e^{2\xi}+1}\right|_{p=1} = \sec h(\xi), \tag{8}$$

$$\cos(w) = \cos\left(w\left(\xi\right)\right) = \left.\frac{p^2e^{2\xi}-1}{p^2e^{2\xi}+1}\right|_{p=1} = \tan h(\xi), \tag{9}$$

where $p$ is the integral constant and non-zero.

After these two major features, as for the definition of SGEM, to get the solution of non-linear partial differential equation as in the form below;

$$P\left(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, u_{xxt}, \ldots\right) = 0, \tag{10}$$

we consider

$$U\left(\xi\right) = \sum_{i=1}^{n}\tanh^{i-1}\left(\xi\right)\left[B_i \sec h\left(\xi\right) + A_i \tanh\left(\xi\right)\right] + A_0. \tag{11}$$

Equation (11) can be rearranged with respect to Equation (8) and Equation (9) as follows;

$$U\left(w\right) = \sum_{i=1}^{n}\cos^{i-1}\left(w\right)\left[B_i \sin\left(w\right) + A_i \cos\left(w\right)\right] + A_0. \tag{12}$$

We implement the balance technique to define the value of $n$ under the highest power non-linear term and highest derivative in the ordinary differential equation. We assume that the summation of coefficients of $\sin^i\left(w\right)\cos^j\left(w\right)$ with the same power is zero, this gives a system of equations. Through software, we solve the system of equations to get the values of $A_i, B_i, \mu$ and $c$. Lastly, substituting the values of $A_i, B_i, \mu$ and $c$ into Equation (11), we obtain the new travelling wave solutions to the Equation (10).

### 3.2. Fundamental Properties of MEFM

In this section the modified $\exp\left(-\Omega(\zeta)\right)$-expansion function method is regulated. Technique [27,30] is a developed form of $\exp\left(-\Omega(\zeta)\right)$-expansion function technique.

Let's think the non-linear partial differential equations to implement this technique as follows;

$$P\left(u, u_x, u_t^{\alpha}, u_{xx}, u_{tt}^{2\alpha}, u_{tx}^{\alpha}, \ldots\right) = 0, \tag{13}$$

where $u = u(x,t)$ is unknown function, $P$ is a polynomial that has $u(x,t)$ function and its partial derivatives respect to $x$ and $t$, $\alpha \in (0,1]$ is the order of the conformable derivative.

**Step 1.** Suppose the traveling wave transformation is

$$u(x,t) = U(\zeta),\ \zeta = x - \frac{lt^{\alpha}}{\alpha}, \tag{14}$$

where $l$ is a non-zero constant that can be defined later. Using partial derivatives of the Equation (14) into Equation (13), the Equation (13) is converted to a non-linear ordinary differential equation defined as;

$$N\left(U, U', U'', U''', \ldots\right) = 0, \tag{15}$$

where $N$ is a polynomial depend on $U$.

**Step 2.** We assume the traveling wave solution of Equation (15) can be phrase as ;

$$U(\zeta) = \frac{\sum_{i=0}^{N} A_i \left[\exp\left(-\Omega(\zeta)\right)\right]^i}{\sum_{j=0}^{M} B_j \left[\exp\left(-\Omega(\zeta)\right)\right]^j} = \frac{A_0 + A_1 \exp(-\Omega) + \ldots + A_N \exp\left(N(-\Omega)\right)}{B_0 + B_1 \exp(-\Omega) + \ldots + B_M \exp\left(M(-\Omega)\right)}, \tag{16}$$

where $A_i, B_j, \left(0 \le i \le N, 0 \le j \le M\right)$ are constants can be defined later, $A_N \neq 0, B_M \neq 0,$ and $\Omega = \Omega(\zeta)$ solves the following ordinary differential equation;

$$\Omega'(\zeta) = \exp\left(-\Omega(\zeta)\right) + \mu \exp\left(\Omega(\zeta)\right) + \lambda. \tag{17}$$

Thinking that we solved Equation (17), we achieve the five solution families as follows [33,34]:

**Family 1:** When $\mu \neq 0, \lambda^2 - 4\mu > 0,$

$$\Omega(\zeta) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\zeta + E)\right) - \frac{\lambda}{2\mu}\right). \tag{18}$$

**Family 2:** When $\mu \neq 0, \lambda^2 - 4\mu < 0,$

$$\Omega(\zeta) = \ln\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}(\zeta + E)\right) - \frac{\lambda}{2\mu}\right). \tag{19}$$

**Family 3:** When $\mu = 0, \lambda \neq 0,$ and $\lambda^2 - 4\mu > 0,$

$$\Omega(\zeta) = -\ln\left(\frac{\lambda}{\exp\left(\lambda(\zeta + E)\right) - 1}\right). \tag{20}$$

**Family 4:** When $\mu \neq 0, \lambda \neq 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Omega(\zeta) = \ln\left(-\frac{2\lambda(\zeta+E)+4}{\lambda^2(\zeta+E)}\right). \tag{21}$$

**Family 5:** When $\mu = 0, \lambda = 0,$ and $\lambda^2 - 4\mu = 0,$

$$\Omega(\zeta) = \ln(\zeta + E) \tag{22}$$

where $A_0, A_1, \ldots, A_N, B_0, B_1, \ldots, B_M, E, \lambda, \mu$ are constants and can be determined later. Using the homogenous balance technique among the highest non-linear terms with the highest order derivatives of $U$ in Equation (16) it can be find a relationship among $N$ and $M$.

**Step 3:** Substituting Equation (17) along with solution families into Equation (16) we have a polynomial of $\exp(\Omega(\zeta))$. After all coefficients of the similar power of $\exp(\Omega(\zeta))$ are equated to zero, returns a system of algebraic equations in terms of $A_0, A_1, \ldots, A_N, B_0, B_1, \ldots, B_M, E, \lambda$. As a result of this process, the obtained values of coefficients substituting into Equation (16), it gives the traveling wave solutions of Equation (13).

## 4. APPLICATIONS OF APPROACHES SGEM AND MEFM

### 4.1. SGEM for the Conformable NLSE Equation in Deep Water

$$i\left(\frac{\partial^\alpha U}{\partial t^\alpha} + c_g \frac{\partial U}{\partial x}\right) - \frac{\omega_0}{8k_0^2}\frac{\partial^2 U}{\partial x^2} - \frac{\omega_0 k_0^2}{2}|U|^2 U = 0, \tag{23}$$

where $\alpha$ is conformable derivative order in $0 < \alpha \leq 1$.

Firstly, we consider the travelling wave transformation as following, for convert the non-linear partial differential equation Equation (23) to a linear ordinary differential equation

$$U(x,t) = \Psi(\zeta)e^{i\varphi}, \quad \zeta = ax - \frac{bt^\alpha}{\alpha}, \varphi = px - \frac{qt^\alpha}{\alpha}, \tag{24}$$

where $a, b, p, q$ are nonzero constants. We have the following corresponding to real part and imaginary part, respectively.

$$\Psi\left(q - c_g p + \frac{p^2\omega_0}{8k_0^2}\right) - \frac{a^2\omega_0}{8k_0^2}\Psi'' - \frac{k_0^2\omega_0}{2}\Psi^3 = 0, \tag{25}$$

$$b = a\left(c_g - \frac{p\omega_0}{4k_0^2}\right). \tag{26}$$

Using homogeneous balance principle between $\Psi''$ and $\Psi^3$ , we get $n = 1$.
We put $n = 1$ into the Equation (12), it gives

$$\Psi(\zeta) = B_1\sin(w) + A_1\cos(w) + A_0. \tag{27}$$

Substituting Equation (27) and its second-order derivative into Equation (25), we obtain a trigonometric function with different degrees. Equating to zero all sum of coefficients of the same power of the trigonometric functions, we obtain an algebraic equation system.

The solution of this algebraic equation system gives the coefficients of Equation (11) i.e $B_1, A_1, A_0$ and $a, b, p, q$.

The graphs of the solutions of Equation (1) with this method are given in Figure 1, 2, 3, 4, 5, 6, 7.

After then we have the following situations:

**Case 1:** $A_0 = 0, A_1 = -iB_1, a = 2\sqrt{2}k_0^2 B_1, c_g = \dfrac{8qk_0^2 + \omega_0 p^2 + 4k_0^4\omega_0 B_1^2}{8pk_0^2},$

$$U_1(x,t) = \Psi_1(x,t)e^{i\varphi(x,t)} = e^{i\varphi}B_1\left(\text{Sech}\left[2\sqrt{2}B_1k_0^2x - \frac{2\sqrt{2}k_0^2B_1}{\alpha}\left(-\frac{p\omega_0}{4k_0^2} + \frac{8qk_0^2 + \omega_0 p^2 + 4k_0^4\omega_0 B_1^2}{8pk_0^2}\right)\right] - i\,\text{Tanh}\left[2\sqrt{2}B_1k_0^2x - \frac{2\sqrt{2}k_0^2B_1t^\alpha}{\alpha}\left(-\frac{p\omega_0}{4k_0^2} + \frac{8qk_0^2 + \omega_0 p^2 + 4k_0^4\omega_0 B_1^2}{8pk_0^2}\right)\right]\right). \tag{28}$$

**Case 2:** $A_0 = 0, A_1 = -\dfrac{ia}{2\sqrt{2}k_0^2}, B_1 = -\dfrac{a}{2\sqrt{2}k_0^2}, q = c_g p - \dfrac{\left(a^2 + 2p^2\right)\omega_0}{16k_0^2},$

$$U_2(x,t) = \Psi_2(x,t)e^{i\varphi(x,t)} = \frac{-a\,e^{i\varphi}\left(\text{Sech}\left[ax - \dfrac{a\left(4k_0^2c_g - p\omega_0\right)t^\alpha}{4k_0^2\alpha}\right] + i\,\text{Tanh}\left[ax - \dfrac{a\left(4k_0^2c_g - p\omega_0\right)t^\alpha}{4k_0^2\alpha}\right]\right)}{2\sqrt{2}k_0^2}. \tag{29}$$

**Case 3:** $A_0 = 0, B_1 = 0, A_1 = -\dfrac{\sqrt{-8c_gk_0^2p + 8k_0^2q + \omega_0 p^2}}{2k_0^2\sqrt{\omega_0}}, a = \dfrac{i\sqrt{-8c_gk_0^2p + 8k_0^2q + \omega_0 p^2}}{\sqrt{2\omega_0}},$

$$U_3(x,t) = \Psi_3(x,t)e^{i\varphi(x,t)} = -ie^{i\varphi}\,\frac{\sqrt{-8c_gk_0^2p + 8k_0^2q + \omega_0 p^2}}{2k_0^2\sqrt{\omega_0}}. \text{Tanh}\left[\frac{x\sqrt{-8c_gk_0^2p + 8k_0^2q + \omega_0 p^2}}{\sqrt{2\omega_0}} - \frac{t^\alpha\left(4k_0^2c_g - p\omega_0\right)\sqrt{-8c_gk_0^2p + 8k_0^2q + \omega_0 p^2}}{4\alpha k_0^2\sqrt{2\omega_0}}\right]. \tag{30}$$

**Case 4:** $A_0 = 0, B_1 = 0, k_0 = \dfrac{(1-i)\sqrt{a}}{2^{3/4}\sqrt{A_1}}, q = c_g p - \dfrac{i\left(2a^2 + p^2\right)\omega_0 A_1}{4\sqrt{2}a},$

$$U_4\left(x,t\right) = \Psi_4\left(x,t\right)e^{i\varphi(x,t)} = e^{i\left(px - \frac{t^\alpha\left(4\sqrt{2}a c_g p - i\left(2a^2+p^2\right)\omega_0 A_1\right)}{4\sqrt{2}a\alpha}\right)} A_1 \operatorname{Tanh}\left[ax - \frac{at^\alpha\left(2\sqrt{2}ac_g - ip\omega_0 A_1\right)}{2\sqrt{2}a\alpha}\right]. \quad (31)$$

**Case 5:** $A_0 = 0, B_1 = iA_1, k_0 = -\dfrac{(-1)^{3/4}\sqrt{a}}{2^{3/4}\sqrt{A_1}}, q = c_g p - \dfrac{i\left(a^2 + 2p^2\right)\omega_0 A_1}{4\sqrt{2}a},$

$$U_5\left(x,t\right) = \Psi_5\left(x,t\right)e^{i\varphi(x,t)} = e^{i\left(px - \frac{qt^\alpha}{\alpha}\right)} A_1 \left( \begin{array}{l} i\operatorname{Sech}\left[ax - \dfrac{at^\alpha\left(\sqrt{2}a c_g - ip\,\omega_0 A_1\right)}{\sqrt{2}a\,\alpha}\right] + \\ \operatorname{Tanh}\left[ax - \dfrac{at^\alpha\left(\sqrt{2}a c_g - ip\omega_0 A_1\right)}{\sqrt{2}a\,\alpha}\right] \end{array} \right). \quad (32)$$

**Case 6:** $A_0 = 0, B_1 = -iA_1, a = 2i\sqrt{2}k_0^2 A_1, \omega_0 = \dfrac{8k_0^2\left(-c_g p + q\right)}{-p^2 + 4k_0^4 A_1^2},$

$$U_6\left(x,t\right) = \Psi_6\left(x,t\right)e^{i\varphi(x,t)} = e^{i\left(px - \frac{qt^\alpha}{\alpha}\right)} A_1 \left( -i\operatorname{Sec}\left[2\sqrt{2}k_0^2 A_1 x - \frac{2\sqrt{2}k_0^2 A_1 t^\alpha\left(c_g - \dfrac{2p\left(-c_g p + q\right)}{-p^2 + 4k_0^4 A_1^2}\right)}{\alpha}\right] + \right.$$
$$\left. i\operatorname{Tan}\left[2\sqrt{2}k_0^2 A_1 x - \frac{2\sqrt{2}k_0^2 A_1 t^\alpha\left(c_g - \dfrac{2p\left(-c_g p + q\right)}{-p^2 + 4k_0^4 A_1^2}\right)}{\alpha}\right]\right). \quad (33)$$

**Case 7:** $A_0 = 0, B_1 = 0, A_1 = \dfrac{ia}{\sqrt{2}k_0^2}, q = c_g p - \dfrac{\left(2a^2 + p^2\right)\omega_0}{8k_0^2},$

$$U_7\left(x,t\right) = \Psi_7\left(x,t\right)e^{i\varphi(x,t)} = \frac{ia}{\sqrt{2}k_0^2} e^{i\left(px - \frac{t^\alpha\left(c_g p - \frac{\left(2a^2+p^2\right)\omega_0}{8k_0^2}\right)}{\alpha}\right)} \operatorname{Tanh}\left[a\,x - \frac{a\,t^\alpha\left(4c_g k_0^2 - p\omega_0\right)}{4k_0^2\alpha}\right]. \quad (34)$$

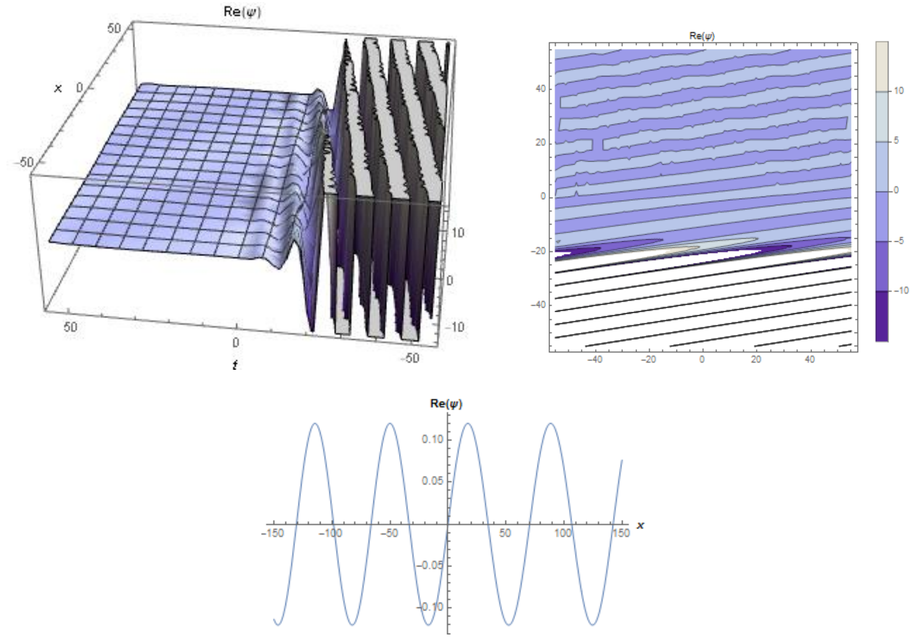



***Figure 1.*** *The 3D and contour graphics of Equation (28) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.1, a = 1.45, B_1 = 0.12, \omega_0 = 0.632456$ *and* $t = 10$

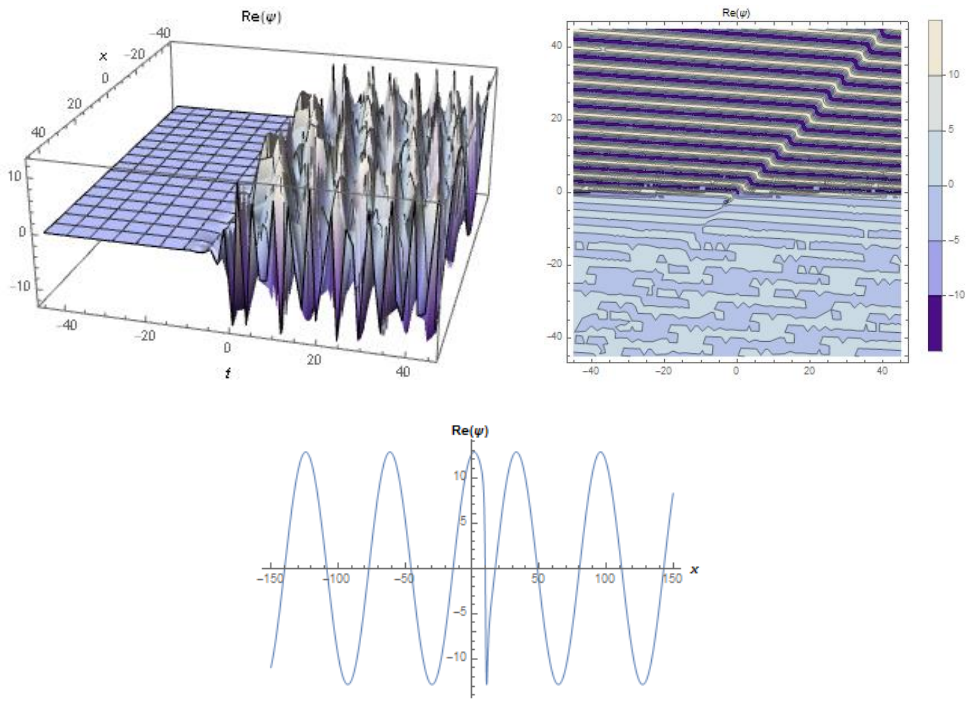



***Figure 2.*** *The 3D and contour graphics of Equation (29) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.1, a = 1.45, \omega_0 = 0.632456$ *and* $t = 10$

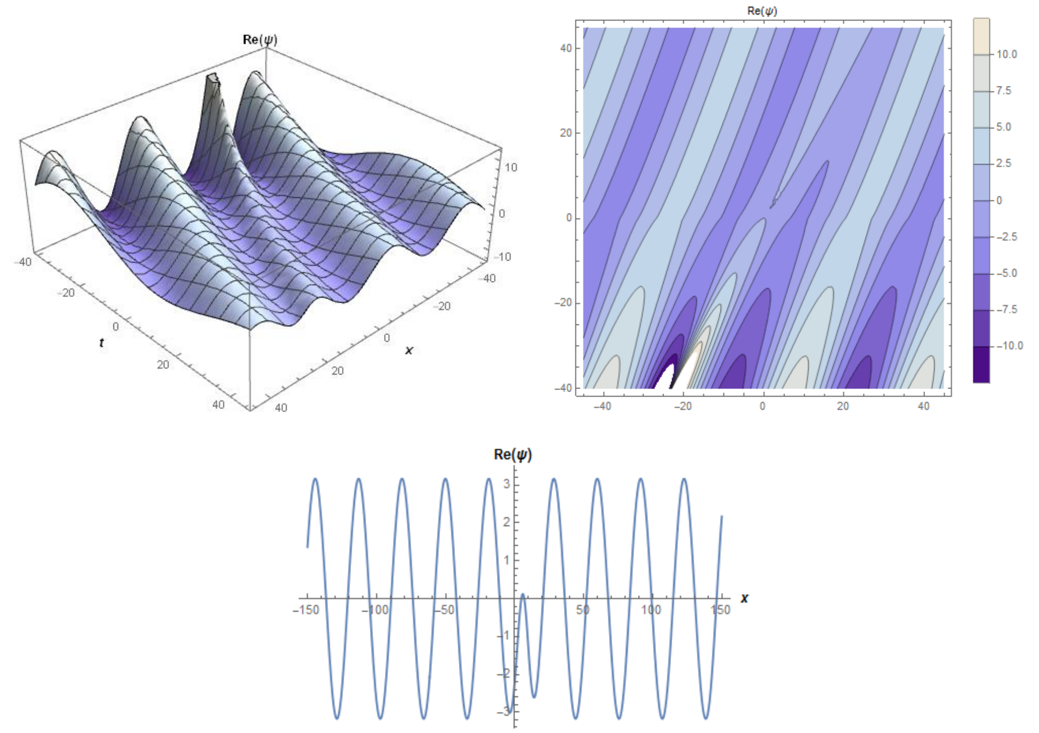



***Figure 3.*** *The 3D and contour graphics of Equation (30) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.2, a = 1.45, \omega_0 = 0.632456$ *and* $t = 10$

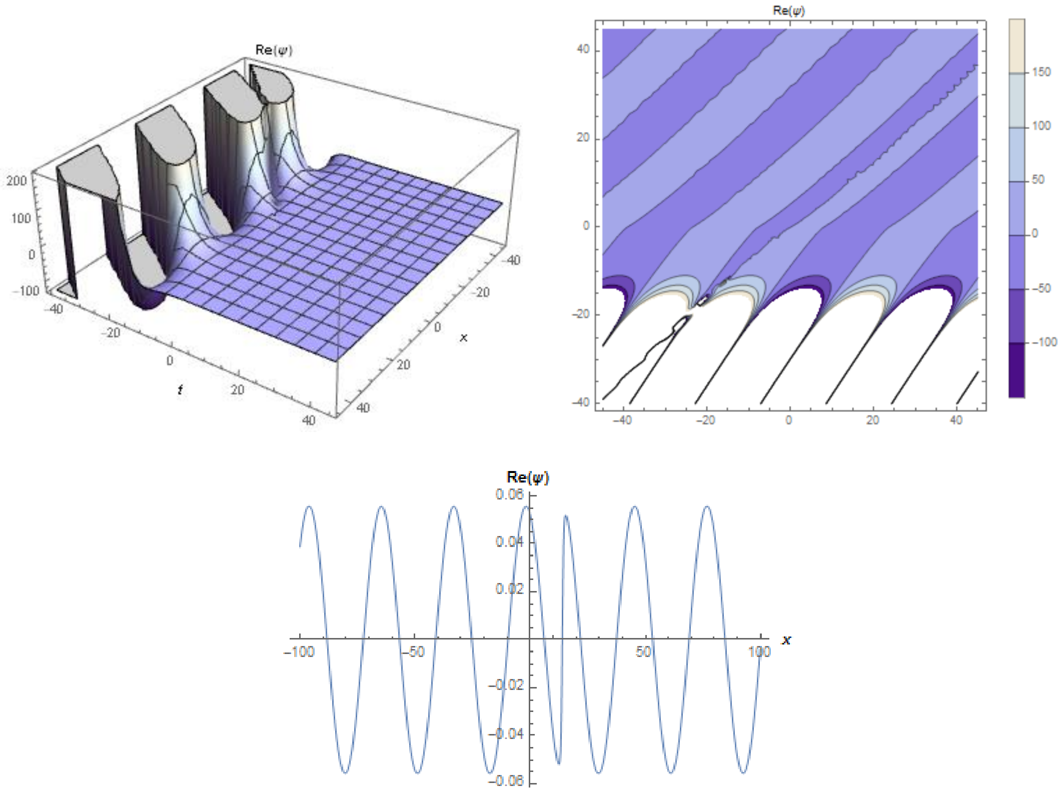



***Figure 4.*** *The 3D and contour plots of Equation (31) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.2, q = 0.11, a = 1.45, \omega_0 = 0.632456, A_1 = 1,$ *and* $t = 10$

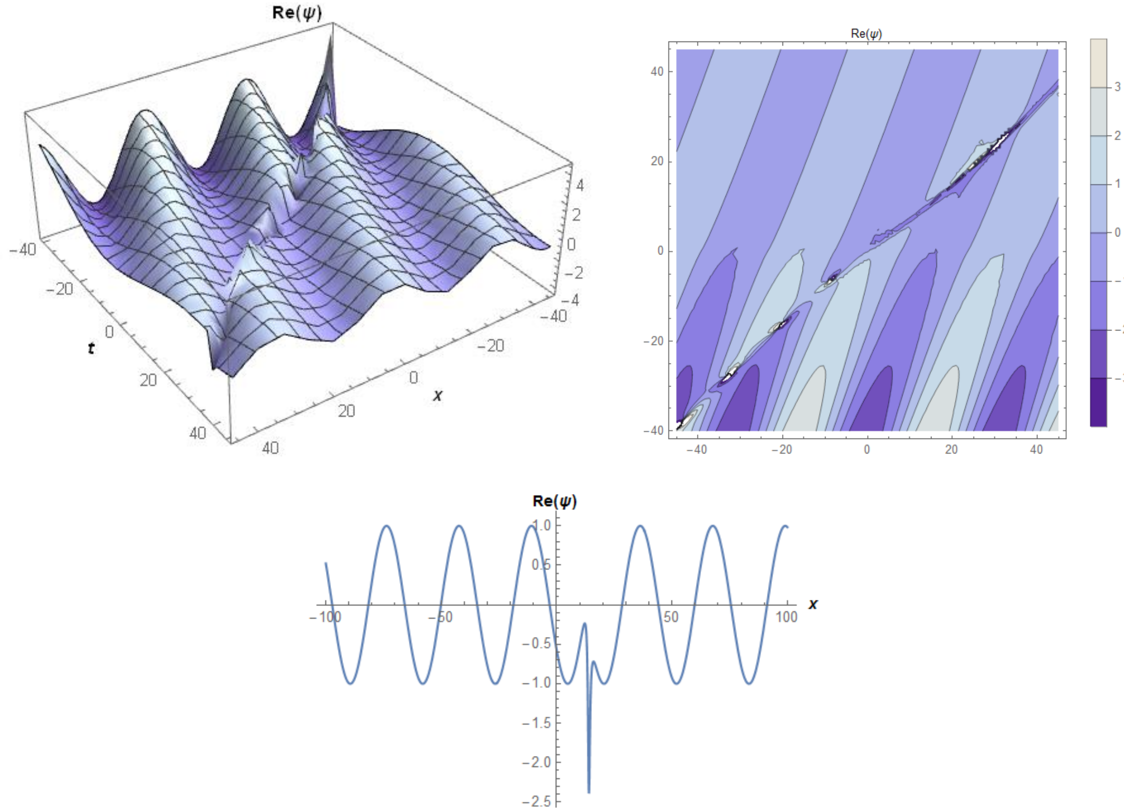
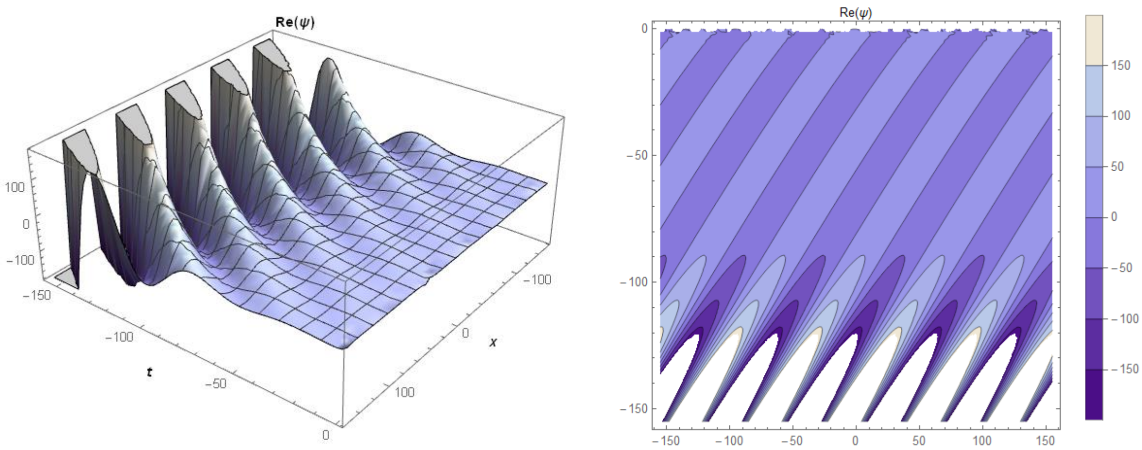



***Figure 5.*** *The 3D and contour graphics of Equation (32) for the values of*
$\alpha = 0.9, k_0 = 0.2, p = 0.2, q = 0.11, a = 1.45, \omega_0 = 0.632456, c_g = 1.58114, A_1 = 1,$ *and* $t = 10$



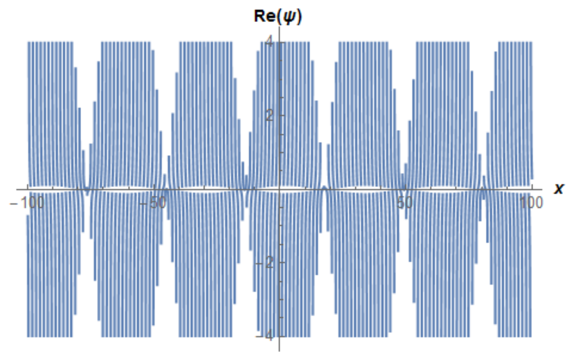



***Figure 6.*** *The 3D and contour graphics of Equation (33) for the values of* $\alpha = 0.9, k_0 = 1.2, p = 0.1, q = 0.2, a = 1.45, \omega_0 = 1.54919, c_g = 0.645497, A_1 = 1,$ *and* $t = 10$

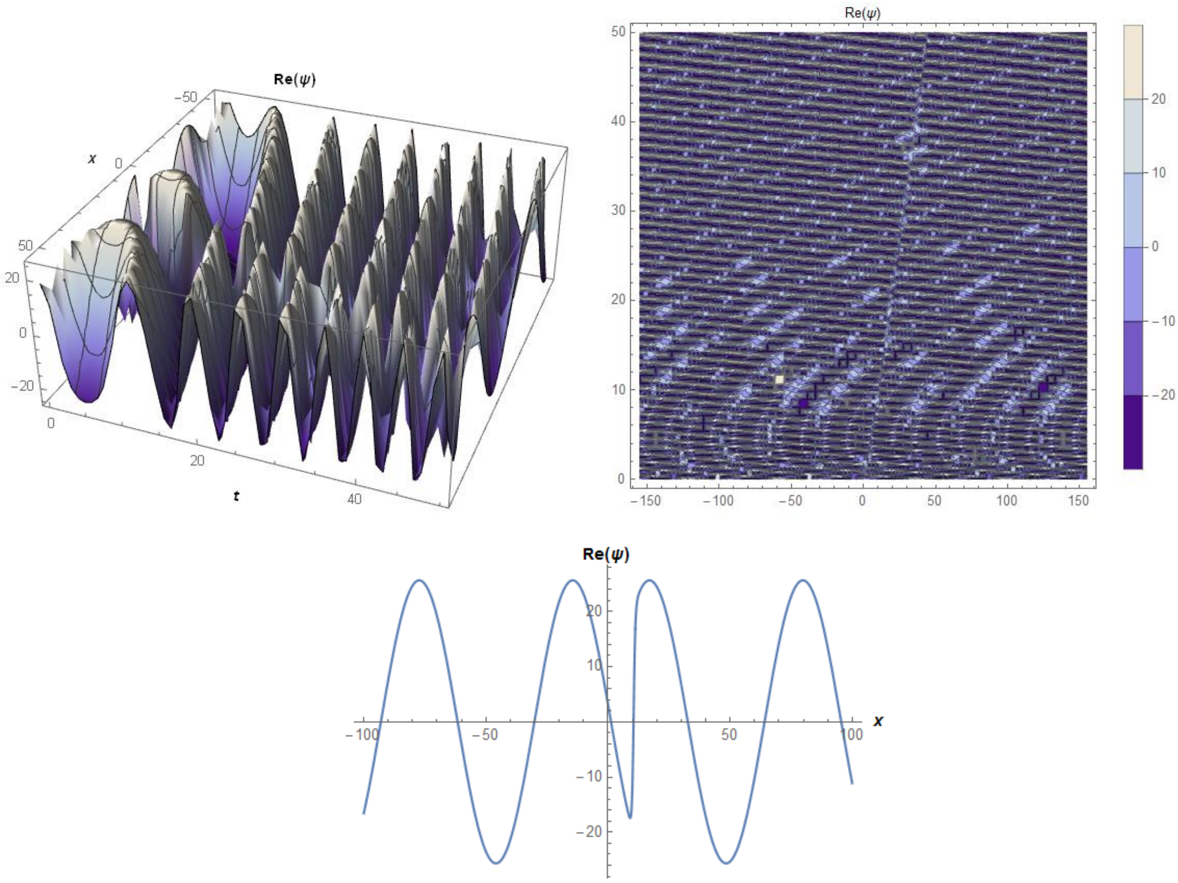



***Figure 7.*** *The 3D and contour graphics of Equation (34) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.1, q = 0.2, a = 1.45, \omega_0 = 0.632456, c_g = 1.58114,$ *and* $t = 10$

### 4.2. MEFM for the Conformable NLSE Equation in Deep Water

In this part, we focus on the soliton solutions Equation (1) by the modified $\exp\left(-\Omega(\zeta)\right)$ -expansion function method.

We use travelling wave transform in Equation (24) as,

$$U(x,t)=\Psi(\zeta)e^{i\varphi}, \quad \zeta=ax-\frac{bt^{\alpha}}{\alpha}, \varphi=px-\frac{qt^{\alpha}}{\alpha}. \tag{35}$$

We have the following non-linear ordinary differential equation corresponding to real part and imaginary part, respectively

$$\Psi\left(q-c_g p+\frac{p^2\omega_0}{8k_0^2}\right)-\frac{a^2\omega_0}{8k_0^2}\Psi''-\frac{k_0^2\omega_0}{2}\Psi^3=0, \tag{36}$$

$$b=a\left(c_g-\frac{p\omega_0}{4k_0^2}\right). \tag{37}$$

Using homogeneous balance principle between $\Psi''$ and $\Psi^3$, we get a connection for $M$ and $N$ as,

$$M+1=N.$$

For appropriate integer values of $M$ and $N$, one can acquire different situations. We have select $M=1$ and $N=2$ values, the solution form as given following yields

$$\Psi(\zeta)=\frac{A_0+A_1e^{-\Omega(\zeta)}+A_2e^{-2\Omega(\zeta)}}{B_0+B_1e^{-\Omega(\zeta)}}. \tag{38}$$

Substituting Equation (38) and its second order derivative into Equation (36), some soliton solutions have emerged as presented.

The graphs of the solutions of Equation (1) with this method are given in Figure 8, 9, 10, 11, 12, 13, 14, 15.

**Case 1:** $A_1=0, B_0=0, \lambda=0, B_1=\frac{i\sqrt{2}k_0^2A_2}{a}, \mu=\frac{A_0}{A_2}, q=c_g p-\frac{p^2\omega_0}{8k_0^2}-\frac{a^2\omega_0A_0}{2A_2k_0^2}.$

Using the coefficients in the upper part, the following solution families are acquired.

**Family 1:**

$$U_{1,1}\left(x,t\right)=\Psi_{1,1}\left(x,t\right)e^{i\varphi(x,t)}=-i\sqrt{\frac{2A_0}{k_0^4A_2}}a\,e^{i\left(px-\frac{\left(c_g p-\frac{p^2\omega_0}{8k_0^2}-\frac{a^2\omega_0A_0}{2A_2k_0^2}\right)t^{\alpha}}{\alpha}\right)}\operatorname{Cosec}\left[\frac{2\sqrt{A_0}\left(E+a\,x-\frac{a\,t^{\alpha}\left(4c_gk_0^2-p\omega_0\right)}{4\alpha k_0^2}\right)}{\sqrt{A_2}}\right] \tag{39}$$

when $\lambda^2-4\mu>0.$

**Family 2:**

$$U_{1,2}\left(x,t\right)=\Psi_{1,2}\left(x,t\right)e^{i\varphi(x,t)}=-i\sqrt{\frac{2A_0}{k_0^4A_2}}a\,e^{i\left(px-\frac{\left(c_g p-\frac{p^2\omega_0}{8k_0^2}-\frac{a^2\omega_0A_0}{2A_2k_0^2}\right)t^{\alpha}}{\alpha}\right)}\operatorname{Cosec}\left[\frac{2\sqrt{A_0}\left(E+a\,x-\frac{a\,t^{\alpha}\left(4c_gk_0^2-p\omega_0\right)}{4\alpha k_0^2}\right)}{\sqrt{A_2}}\right] \tag{40}$$

when $\lambda^2-4\mu<0.$

**Family 5:**

$$U_{1,5}\left(x,t\right)=\Psi_{1,5}\left(x,t\right)e^{i\varphi(x,t)}=-\frac{i\,2\sqrt{2}\,a\,\alpha\,e^{i\left(px-\frac{\left(8k_0^2c_g p-p^2\omega_0\right)t^{\alpha}}{8k_0^2\,\alpha}\right)}}{4k_0^4\left(E+a\,x\right)\alpha+at^{\alpha}\left(-4c_gk_0^2+p\omega_0\right)}, \tag{41}$$

when $\mu=0,\lambda=0,$ and $\lambda^2-4\mu=0.$

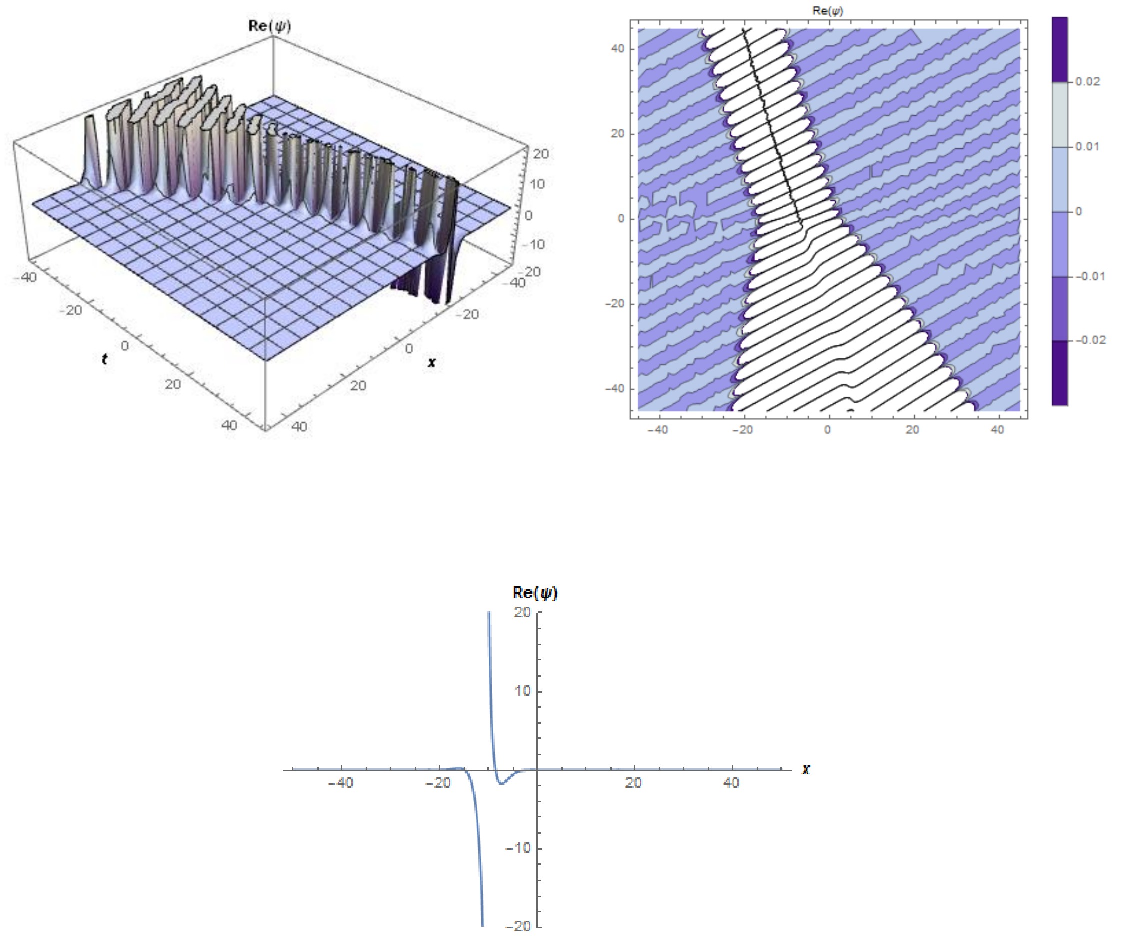
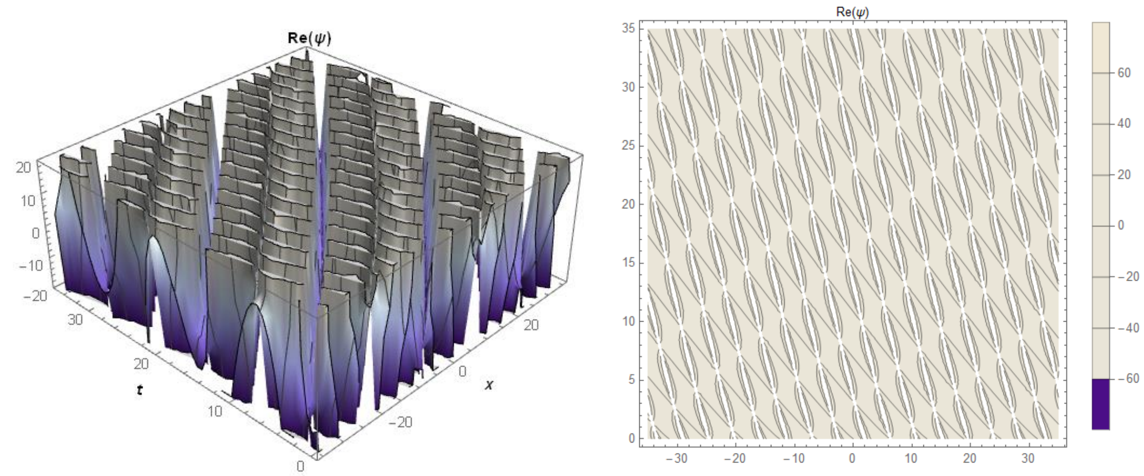



***Figure 8.*** *The 3D, 2D and contour graphics of Equation (39) for the values of*
$\alpha = 0.9, k_0 = 0.2, p = 0.5, q = 1.2, a = 1.45, \omega_0 = 0.632456, c_g = 1.58114, A_0 = -0.12, A_2 = 2, E = 10$ *and* $t = 10$



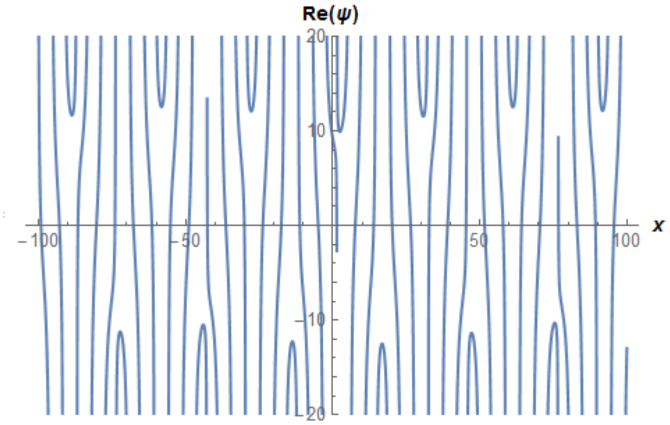



***Figure 9.*** *The 3D, 2D and contour graphics of Equation (40) for the values of* $\alpha=0.9, k_0=0.2, p=0.5, q=1.2, a=1.45, \omega_0=0.894427, c_g=2.23607, A_0=-0.12, A_2=2, E=10$ *and* $t=10$

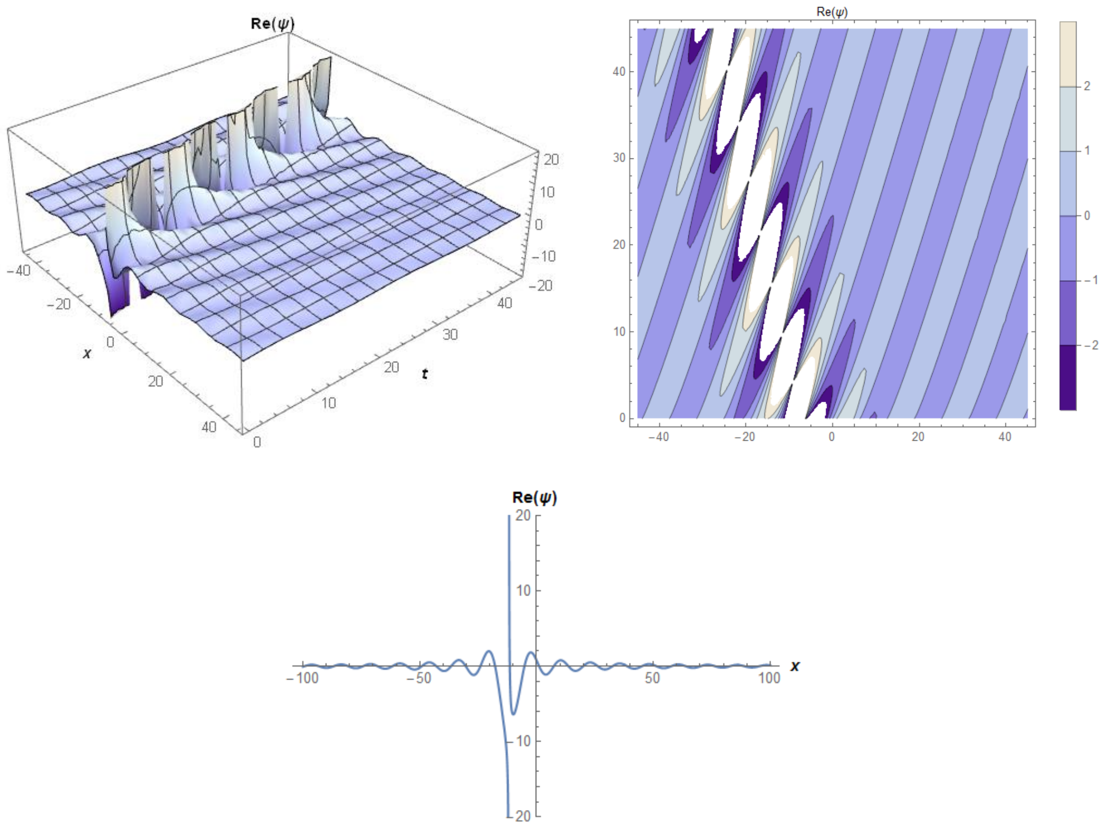



***Figure 10.*** *The 3D, 2D and contour graphics of Equation (41) for the values of* $\alpha=0.9, k_0=0.2, p=0.5, q=1.2, a=1.45, \omega_0=0.894427, c_g=2.23607, E=10$ *and* $t=10$

**Case 2:**

$$A_0=-\frac{1}{4}\lambda^2 A_2, A_1=0, B_1=-\frac{i\sqrt{2}k_0^2 A_2}{a}, B_0=\frac{ik_0^2\lambda A_2}{a\sqrt{2}}, c_g=\frac{16k_0^2 q+2p^2\omega_0+a^2\omega_0\left(\lambda^2-4\mu\right)}{16k_0^2 p},$$

Using the coefficients in the upper part, the following solution families are acquired.

**Family 1:**

$$U_{2,1}(x,t)=\Psi_{2,1}(x,t)e^{i\varphi(x,t)}=ia\,e^{i\left(px-\frac{qt^{\alpha}}{\alpha}\right)}\times$$

$$\left(\lambda^2-4\mu+\lambda\sqrt{\lambda^2-4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\left(E+ax-\frac{at^{\alpha}\left(16k_0^2q-2p^2\omega_0+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}{16k_0^2p\alpha}\right)\right]\right)/ \qquad (42)$$

$$2\sqrt{2}k_0^2\left(\lambda+\sqrt{\lambda^2-4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\left(E+ax-\frac{at^{\alpha}\left(16k_0^2q-2p^2\omega_0+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}{16k_0^2p\alpha}\right)\right]\right)$$

when $\lambda^2-4\mu>0$.

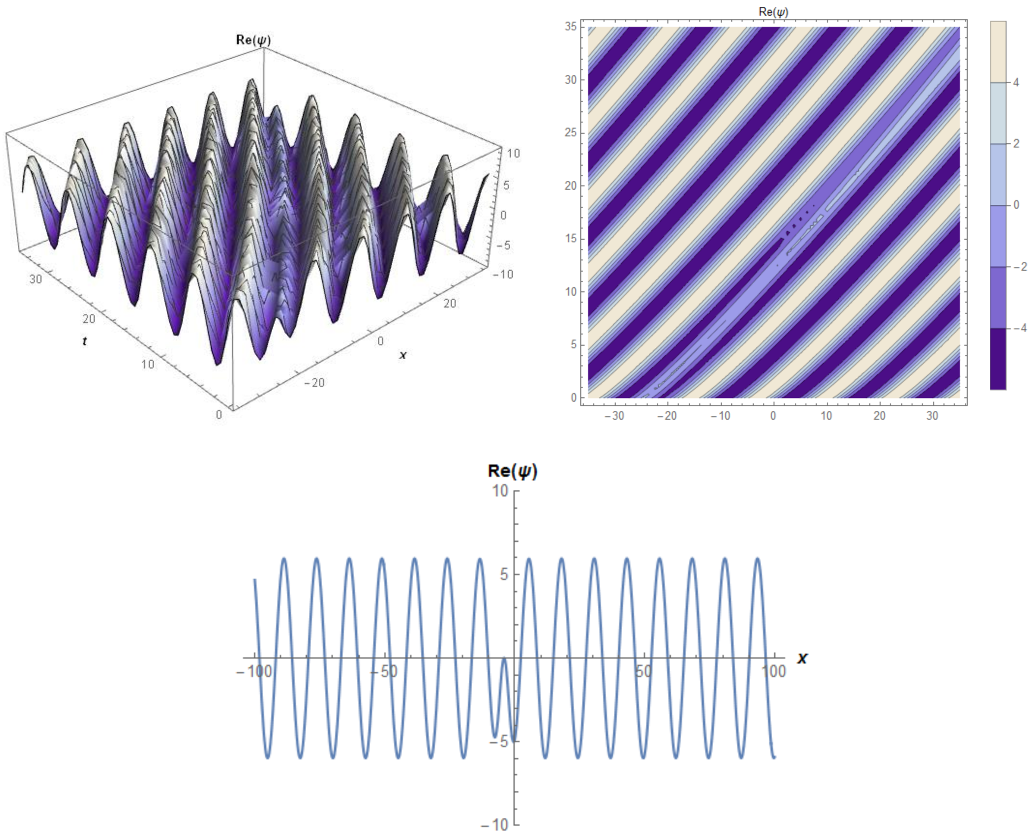



***Figure 11.*** *The 3D, 2D and contour graphics of Equation (42) for the values of* $\alpha=0.9, k_0=0.2, p=0.5, q=1.2, a=0.45, \omega_0=0.894427, \lambda=2.5, \mu=1, E=10$ *and* $t=10$

**Family 2:**

$$U_{2,2}(x,t)=\Psi_{2,2}(x,t)\times e^{i\varphi(x,t)}=ia\,e^{i\left(px-\frac{qt^{\alpha}}{\alpha}\right)}\times$$
$$\left(\lambda^2-4\mu-\lambda\sqrt{-\lambda^2+4\mu}\,\mathrm{Tan}\left[\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(E+ax-\frac{at^{\alpha}\left(16k_0^2q-2p^2\omega_0+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}{16k_0^2p\alpha}\right)\right]\right)/$$

$$2\sqrt{2}k_0^2\left(\lambda-\sqrt{-\lambda^2+4\mu}\,\mathrm{Tan}\left[\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(E+ax-\frac{at^{\alpha}\left(16k_0^2q-2p^2\omega_0+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}{16k_0^2p\alpha}\right)\right]\right), \tag{43}$$

when $\lambda^2-4\mu<0.$

**Family 3:**

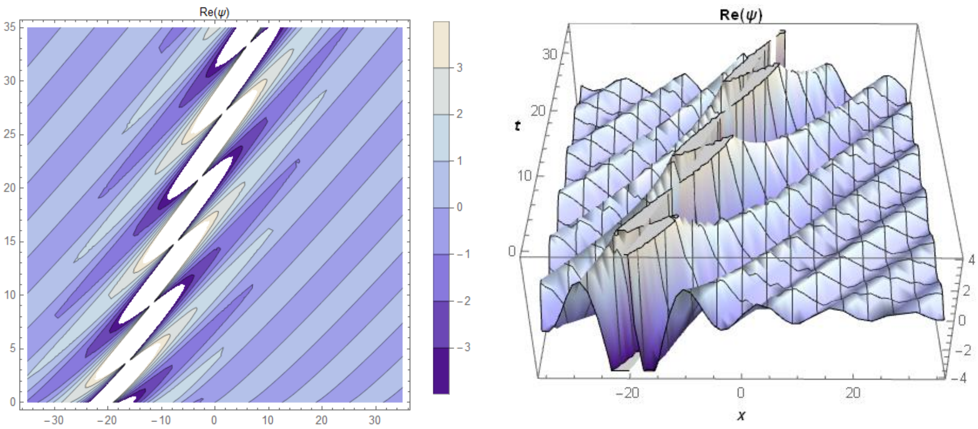
$$U_{2,3}(x,t)=\Psi_{2,3}(x,t)e^{i\varphi(x,t)}=\frac{ia\lambda\,e^{i\left(px-\frac{qt^{\alpha}}{\alpha}\right)}}{2\sqrt{2}k_0^2}\left(1+2\left(-1+e^{\lambda\left(E+ax-\frac{at^{\alpha}\left(16k_0^2q-2p^2\omega_0+a^2\lambda^2\omega_0\right)}{16\alpha k_0^2p}\right)}\right)^{-1}\right), \tag{44}$$

when $\mu=0,\lambda\neq 0,$ and $\lambda^2-4\mu>0.$

**Family 4:**

$$U_{2,4}(x,t)=\Psi_{2,4}(x,t)e^{i\varphi(x,t)}=-\frac{4i\sqrt{2\mu}\,p\,a\alpha\,e^{i\left(px-\frac{qt^{\alpha}}{\alpha}\right)}}{8k_0^2p\alpha\left(-1+\sqrt{\mu}\left(E+ax\right)\right)-at^{\alpha}\sqrt{\mu}\left(8k_0^2q-p^2\omega_0\right)}, \tag{45}$$

when $\mu\neq 0,\lambda\neq 0,$ and $\lambda^2-4\mu=0.$



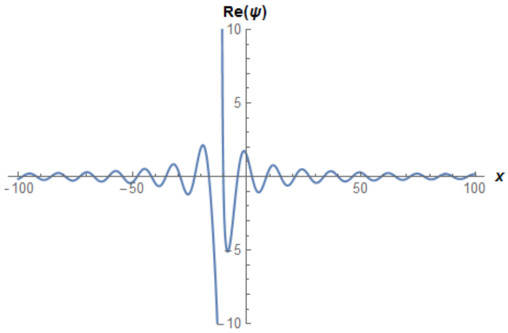



***Figure 12.*** *The 3D, 2D and contour graphics of Equation (45) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.5, q = 1.2, a = 0.45, \omega_0 = 0.894427, \lambda = 2.5, \mu = 1, E = 10$ *and* $t = 10$

**Family 5:**

$$U_{2,5}(x,t) = \Psi_{2,5}(x,t)e^{i\varphi(x,t)} = \frac{4i\sqrt{2}\,p\,a\alpha\, e^{i\left(px - \frac{qt^{\alpha}}{\alpha}\right)}}{8k_0^2 p\alpha(E+ax) + at^{\alpha}\left(-8k_0^2 q + p^2\omega_0\right)}, \tag{46}$$

when $\mu = 0, \lambda = 0,$ and $\lambda^2 - 4\mu = 0.$

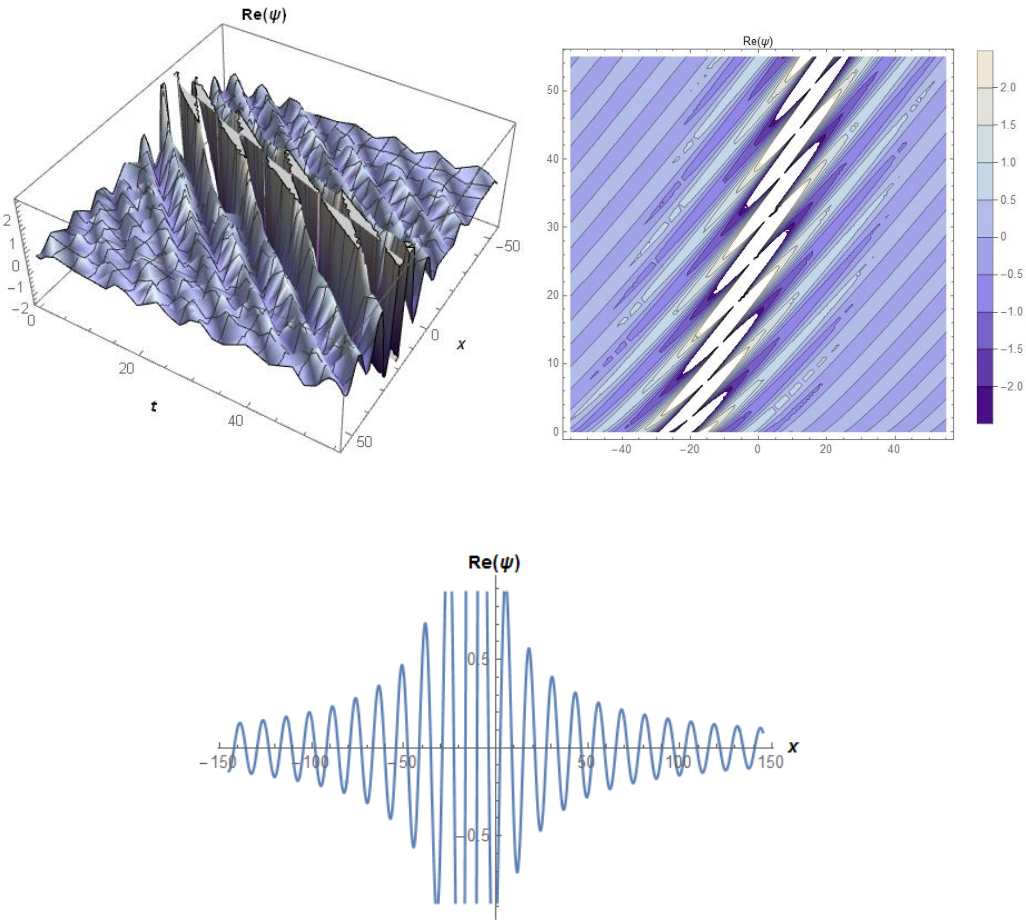



***Figure 13.*** *The 3D, 2D and contour graphics of Equation (46) for the values of* $\alpha = 0.9, k_0 = 0.2, p = 0.5, q = 1.2, a = 0.45, \omega_0 = 0.894427, E = 10$ *and* $t = 10$

**Case 3:**

$$A_0=\frac{\lambda A_1}{4}, A_2=\frac{A_1}{\lambda}, B_0=\frac{ik_0^2A_1}{a\sqrt{2}}, B_1=\frac{i\sqrt{2}k_0^2A_1}{a\lambda}, p=\frac{8c_gk_0^2+\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}}{2\omega_0},$$

Using the coefficients in the upper part, the following solution families are acquired.

**Family 1:**

$$U_{3,1}\left(x,t\right)=\Psi_{3,1}\left(x,t\right)e^{i\varphi(x,t)}=-ia\,e^{i\left(px-\frac{qt^\alpha}{\alpha}\right)}\times$$
$$\left(\lambda^2-4\mu+\lambda\sqrt{\lambda^2-4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{\lambda^2-4\mu}}{16k_0^2\alpha}\left(8k_0^2\alpha E+8a\,\alpha k_0^2x+at^\alpha\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}\right)\right]\right)/$$

$$2\sqrt{2}k_0^2\left(\lambda+\sqrt{\lambda^2-4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{\lambda^2-4\mu}}{16k_0^2\alpha}\left(8k_0^2\alpha E+8a\,\alpha k_0^2x+at^\alpha\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}\right)\right]\right),\tag{47}$$

when $\lambda^2-4\mu>0$.

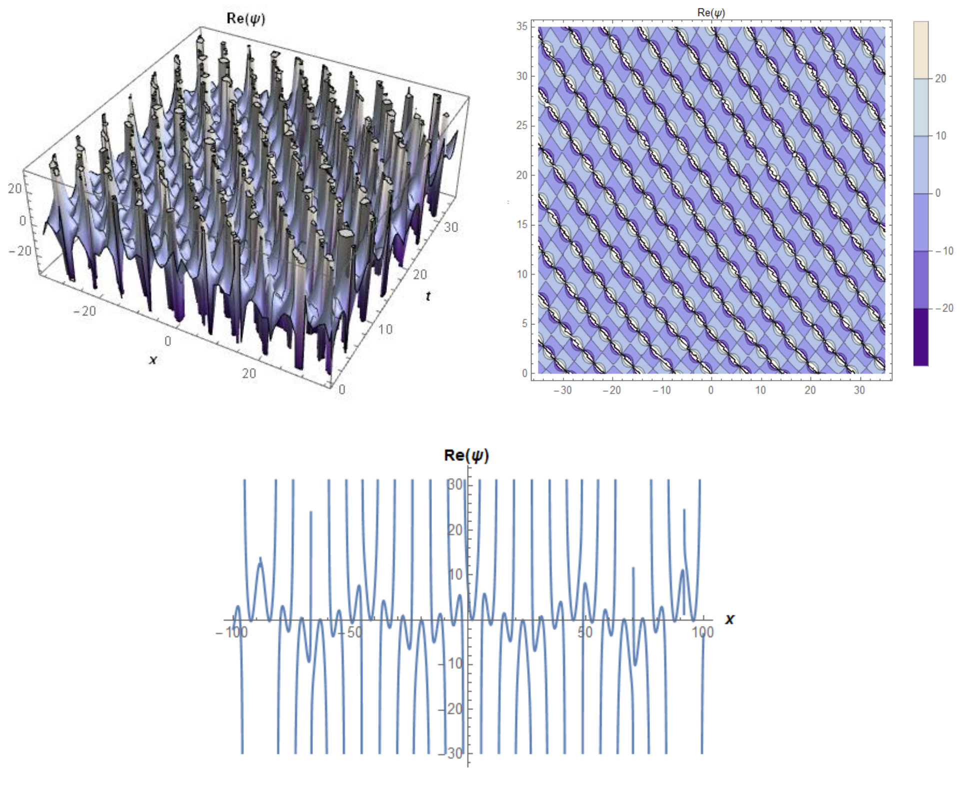



***Figure 14.*** *The 3D, 2D and contour graphics of Equation (47) for the values of* $\alpha=0.9, k_0=0.2, p=0.5, q=1.2, a=0.45, \omega_0=0.894427, c_g=2.23607, \lambda=0.5, \mu=1, E=10$ *and* $t=10$

**Family 2:**

$$U_{3,2}\left(x,t\right)=\Psi_{3,2}\left(x,t\right)e^{i\varphi(x,t)}=-ia\,e^{i\left(px-\frac{qt^{\alpha}}{\alpha}\right)}\times$$

$$\left(\lambda^2-4\mu-\lambda\sqrt{-\lambda^2+4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{-\lambda^2+4\mu}}{16k_0^2\alpha}\left(8k_0^2\alpha E+8a\,\alpha k_0^2x+at^{\alpha}\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}\right)\right]\right)/$$

$$2\sqrt{2}k_0^2\left(\lambda-\sqrt{-\lambda^2+4\mu}\,\mathrm{Tanh}\left[\frac{\sqrt{-\lambda^2+4\mu}}{16k_0^2\alpha}\left(8k_0^2\alpha E+8a\,\alpha k_0^2x+at^{\alpha}\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\omega_0\left(\lambda^2-4\mu\right)\right)}\right)\right]\right), \tag{48}$$

when $\lambda^2-4\mu<0.$

**Family 3:**

$$U_{3,3}\left(x,t\right)=\Psi_{3,3}\left(x,t\right)e^{i\varphi(x,t)}=-\frac{ia\lambda}{2\sqrt{2}k_0^2}e^{i\left(-\frac{qt^{\alpha}}{\alpha}+\frac{x\left(8c_gk_0{}^2+\sqrt{64c_g^2k_0^4-2w_0\left(16k_0{}^2q+a^2\lambda^2w_0\right)}\right)}{2w_o}\right)}$$

$$\mathrm{Coth}\left[\frac{\lambda\left(8k_0^2\alpha E+8a\,\alpha k_0^2x+at^{\alpha}\sqrt{64c_g^2k_0^4-2\omega_0\left(16qk_0^2+a^2\lambda^2\omega_0\right)}\right)}{16k_0^2\alpha}\right], \tag{49}$$

when $\mu=0,\lambda\neq 0,$ and $\lambda^2-4\mu>0.$

**Family 4:**

$$U_{3,4}\left(x,t\right)=\Psi_{3,4}\left(x,t\right)e^{i\varphi(x,t)}=\frac{i\sqrt{2\mu}\,a\alpha\,e^{i\left(\frac{x\left(8c_gk_0^2+\sqrt{64c_g^2k_0^4-32qk_0^2\omega_0}\right)}{2\omega_0}-\frac{qt^{\alpha}}{\alpha}\right)}}{2k_0^2\alpha\left(-1+\sqrt{\mu}\left(E+ax\right)\right)+at^{\alpha}\sqrt{2\mu}\left(2c_g^2k_0^4-k_0^2q\omega_0\right)}, \tag{50}$$

when $\mu\neq 0,\lambda\neq 0,$ and $\lambda^2-4\mu=0.$

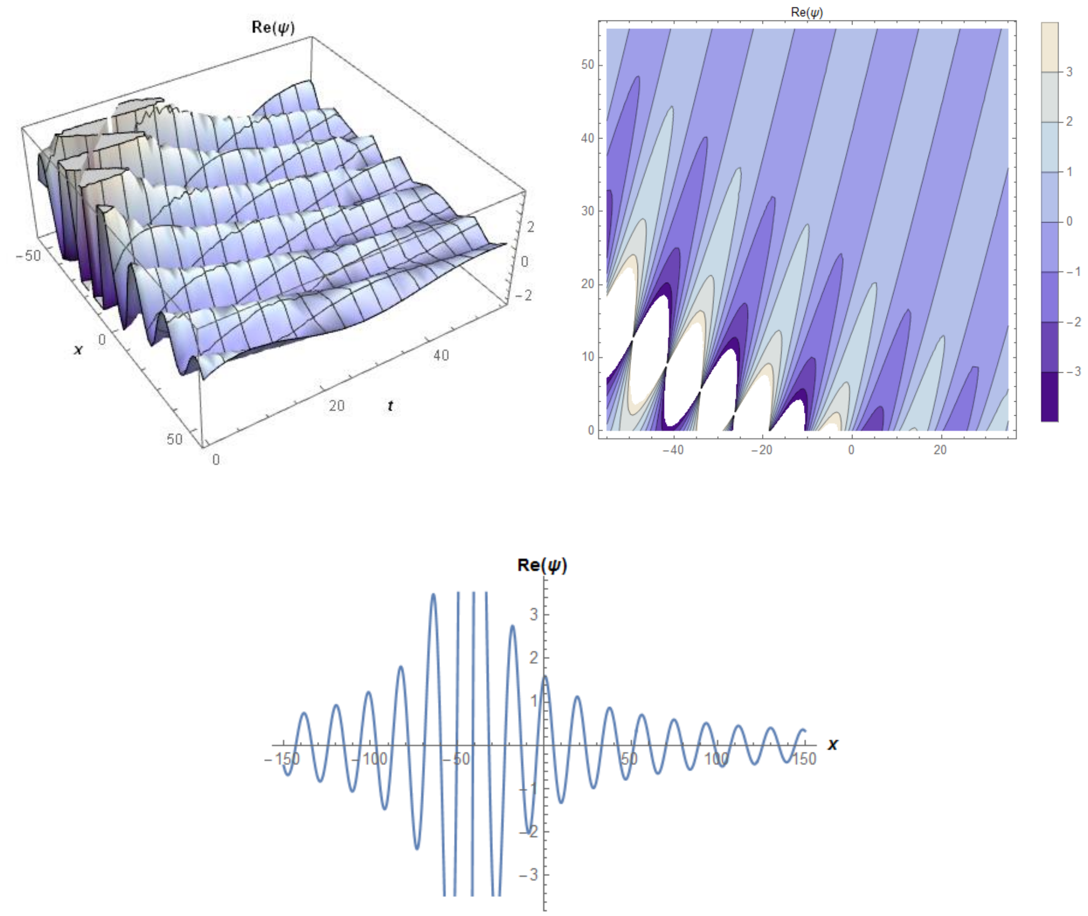



***Figure 15.*** *The 3D, 2D and contour graphics of Equation (50) for the values of* $\alpha = 0.9, k_0 = 0.1, p = 0.5, q = 0.2, a = 0.45, \omega_0 = 0.774597, c_g = 3.87298, \mu = 1, E = 10$ *and* $t = 10$

## 5. RESULTS

We mentioned two analytic methods which are the SGEM and the modified $\exp(-\Omega(\zeta))$-expansion function technique to find out different types of soliton solutions to the NLSE describing gravity waves in deep water. We use the definition of the conformable derivative in calculations. We have drawn the 2D-3D and contour surfaces under the appropriate values of constants. When we check against the acquired solutions with [25, 35, 36, 37], we observe that all solutions obtained corresponding to experimental results. For this reason, we think that providing more calculation convenience numerically of submitted soliton solutions may be much useful particularly in engineering fields. In addition, the recommended methods are very efficient and easy to application non-linear differential models such as governing Equation (1).

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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