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Research Article

Distributed Control of a Vibrating String in Response to Pointwise Force Application

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ABSTRACT

The problem of controlling the vibrations of a string by a discrete applied force is considered. The vibrations of the string are modeled by the linear wave equation and the control is provided by an added force term. The wave equation is solved for controlled and uncontrolled cases with and without control force term. The applied force is chosen to be proportional to string displacement at some specified point. In the controlled case; the wave equation involves a control parameter (gain) and related terms involving the value of the displcement at a single point and a delta function. This makes the equation quite different from the usual wave equation. The problem is solved analytically using a modified (compared to usual wave equation) solution procedure and an equation relating the string eigenfrequencies to the proportionality constant (gain) is derived. This allows the observation of the change in eigenfrequencies with the gain. Finally, examples of uncontrolled and controlled responses are presented, graphically. The results show that the resonances can be avoided by the applied control procedure.

1. Introduction

Distributed control conceivably has important applications in many areas including aerospace technology. For example, the problem of controlling the panel flutter is receiving attention due to its occurrence at high speeds. The specific difficulty of distributed control comes from its being modeled by partial differential equations. One way of circumventing this is to discretize the system by some numerical method and apply the well-known methods of lumped-parameter control. This, however, has the danger of losing some of the physics in the problem.

Pointwise control refers to the measurement and actuation processes being performed at certain points (and not at continuous intervals) within the problem domain. This type of control design is more realistic to be implemented in real-life problems. Several problems relating to the pointwise control on elastic structures, parabolic equations and the wave equation were considered by You [1], Sadek [2], Wang [3], Cherid et al. [4], Droniou and Raymond [5], Sadek et al. [6], Guo and Xie [7], Beauchard [8] and Ouzahra [9]. Nguyen and Raymond [10] attempted to apply pointwise control concept to a fluid mechanics problem. Recently, Sirota and Halevi [11] investigated the control of a membrane (two-dimensional wave equation) using Laplace transforms and transfer functions. In a more recent study, Latas [12] investigated the suppression of the waves traveling along a moving string by distributed force.

In this article, we will consider a control problem related to the one-dimensional wave equation which might be thought of as governing the vibrations of a string (among other things). Measurement and actuation will be performed on two points on the string, which may be identical. The controlled wave equation will involve one (or possibly more) control term that includes the

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value of the physical variable (displacement of string) to be controlled, measured at one point.

Therefore, the resulting mathematical problem (Eq.(5)) is not the usual wave equation since it involves the value of the displacement at one point combined with a delta function as an extra term. To solve this problem, the usual solution procedure of expanding the displacement field in terms of the eigenfunctions of the spatial part of the problem will be modified as explained in the next section. The controlled problem still has eigenfrequencies and eigenfunctions, but the eigenfunctions may no longer be orthogonal. In that case, the eigenfunctions will be orthogonalized using the Gram-Schmit procedure. The controlled eigenfrequencies are also different from the usual wave equation eigenfrequencies, and the values of the controlled eigenfrequencies can be modified by inserting a control parameter (a constant, gain) into the control term. This procedure makes it possible to avoid any possible resonances by simply changing the control parameter.

The paper is organized as follows. In Chapter 2, the problem description and the response of the string both with and without the control term are given. In Chapter 3, the response of the string under an external force, both without the control term and with the control term are presented. Finally, in Chapter 4, graphical solutions for all mentioned cases will be provided.

2. Problem Formulation and Solution

Consider a string of length one pinned at both ends, at $x = 0$ and $x = 1$ as shown in Figure 1.

We assume that the problem is suitably nondimensionalized so we do not have to worry about certain parameters like the length of the string, material properties, and applied forces.

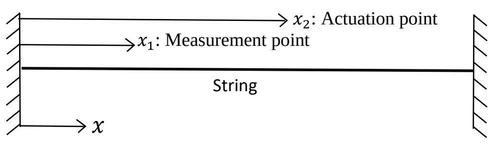

Figure 1. Geometry of the structure.

The small vibrations of the string are governed by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

where $u(x,t)$ denotes the displacement of the string. We set the wave velocity equal to one because of nondimensionalisation. The boundary conditions at the pinned ends are

$$u(0,t) = u(1,t) = 0 \qquad (2)$$

To find the eigenfrequencies, one makes the substitution

$$u(x,t) = U(x)e^{i\omega t} \qquad (3)$$

and this gives the eigenfrequencies and the mode shapes as

$$\omega_n = n\pi,\ U_n(x) = \sin n\pi x,\ n = 1,2,3,\ldots \qquad (4)$$

We apply control to this problem in the following manner: displacement u is measured at some point x_1, at all times, and a force proportional to $u(x_1,t)$ is applied at another point x_2. Thus the vibration equation in the controlled case becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + K\, u(x_1,t)\delta(x - x_2) = 0 \qquad (5)$$

where δ is the Dirac delta function, and K is a constant which can be considered as a gain. We want to investigate the changes in the system as K is changed. In passing, we note that a related problem

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + K\, \frac{\partial U}{\partial t}(x_1,t)\, \delta(x - x_2) = 0 \qquad (6)$$

in which the control is proportional to the velocity, rather than displacement, was considered in [13-15]. For this problem, the total energy of the string

$$E = \frac{1}{2}\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial U}{\partial x}\right)^2\right] dx$$

satisfies

$$\frac{dE}{dt} = -\left[\frac{\partial U}{\partial t}(x_1,t)\right]^2 < 0 \qquad (7)$$

Thus, total energy decreases and therefore the control term in Eq. (6) stabilizes the string. However, in the present problem (Eq.(5)) the control is proportional to displacement rather than velocity and no result similar to Eq. (7) is known. Thus, it is important to investigate the solution of Eq.(5). Again, making the substitution $u(x,t) = U(x)e^{i\omega t}$,

$$\frac{d^2U}{dx^2} + \omega^2 U + K\, U(x_1)\, \delta(x - x_2) = 0 \tag{8}$$

This equation must be solved with the boundary conditions

$$U(0) = U(1) = 0 \tag{9}$$

Since $U(x_1)$ is (an unknown) constant, assuming

$$U(x) = C_1(x)\cos\omega x + C_2(x)\sin\omega x \tag{10}$$

where $C_1(x)$ and $C_2(x)$ are functions to be found, we obtain the solution satisfying the boundary conditions as

$$U(x) = \frac{K}{\omega}\, U(x_1)\left[\sin\omega(x_2 - x)\ h(x - x_2) + \frac{\sin\omega(1-x_2)}{\sin\omega}\sin\omega x\right] \tag{11}$$

where h is the unit step function. Setting $x = x_1$ in the above solution gives the eigenvalue equation,

$$1 = \frac{K}{\omega}\left[\sin\omega(x_2 - x_1)\ h(x_1 - x_2) + \frac{\sin\omega(1-x_2)}{\sin\omega}\sin\omega x_1\right] \tag{12}$$

Roots ω_n of this equation are eigenvalues with the corresponding eigenfunctions given by Eq.(11), ignoring the factor in front,

$$U_n(x) = \sin\omega_n(x_2 - x)\ h(x - x_2) + \frac{\sin\omega_n(1-x_2)}{\sin\omega_n}\sin\omega_n x \tag{13}$$

From Eqs. (8) and (9), we obtain

$$\int_0^1 U_m(x)U_n(x)dx = K\ \frac{U_m(x_1)U_n(x_1)}{\omega_n^2 - \omega_m^2}\left(\frac{\sin\omega_n x_2}{\sin\omega_n x_1} - \frac{\sin\omega_m x_2}{\sin\omega_m x_1}\right) \text{ for } n \neq m \tag{14}$$

so that $U_n(x)$ are not orthogonal unless measurement and actuation are at the same point, i.e., $x_1 = x_2$. To clarify further discussion, assume that $x_1 < x_2$ thus $h(x_1 - x_2) = 0$ and the eigenvalue equation becomes

$$\omega\sin\omega - K\ \sin\omega x_1 \sin\omega(1 - x_2) = 0 \tag{15}$$

For $K = 0$ this gives the same eigenvalues as in Eq.(4), the uncontrolled case, while for $K = \infty$, the eigenvalues are

$$\omega_n = \frac{n\pi}{x_1}\ ,\omega_m = \frac{m\pi}{1-x_2}\quad ,\quad m,n = 1,2,3,\ldots \tag{16}$$

Eigenvalues change smoothly between Eqs.(4) and (16) as K takes values between 0 and ∞. This allows us to shape the output of the system.

3. Uncontrolled and Controlled Response

In this section we will compute the response of the string to an external force with and without the control action under zero initial conditions. For the uncontrolled string

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \cos\pi t\sin\pi x \tag{17}$$

with the boundary and initial conditions

$$u(0,t) = u(1,t) = 0,\quad u(x,0) = u_t(x,0) = 0 \tag{18}$$

The solution of Eqs. (17) and (18) is:

$$u(x,t) = \frac{1}{4\pi}[\cos\pi t - \cos 3\pi t - 2t\sin\pi t]\sin\pi x \tag{19}$$

The presence of $\cos\pi t$ in the external forcing term (Eq.(17)) causes a resonance and the response, Eq.(19), blows up with time. The aim of control, in this case, will be to suppress the unboundedly growing vibrations of the string. The controlled response is governed by

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + K\, u(x_1,t)\delta(x - x_2) = \cos\pi t\sin\pi x \tag{20}$$

with the same conditions Eq.(18). We will take the gain constant as $K = 2$ and examine the response for two different cases of measurement and actuation:

(1) Measurement and actuation are at the same point: $x_1 = x_2 = 1/3$

In this case the eigenfunctions, Eq.(13), are orthogonal; expanding the solution as

$$u(x,t) = \sum_{n=0}^{\infty} p_n(t) U_n(x) \tag{21}$$

we find that the time factors satisfy

$$\frac{d^2 p_n}{dt^2} + \omega_n^2 p_n = -\alpha_n \cos \pi t \tag{22}$$

$$p_n(0) = \frac{dp_n}{dt}(0) = 0 \tag{23}$$

where

$$\alpha_n = \frac{\int_0^1 \sin \pi x \, U_n(x) dx}{\int_0^1 [U_n(x)]^2 dx} \tag{24}$$

The solution is, unless $\omega_n = \pi$,

$$p_n(t) = \frac{\alpha_n}{\omega_n^2 - \pi^2} (\cos \omega_n t - \cos \pi t) \tag{25}$$

(2) Measurement and actuation are at different points: $x_1 = 1/3$, $x_2 = 2/3$

In this case the eigenfunctions are not orthogonal. But we can orthogonalize them using the Gram-Schmidt procedure. Naming the orthogonalized eigenfunctions $\varphi_n(x)$; these satisfy the same Eqs.(8) and (9). Expanding the solution as

$$u(x,t) = \sum_{n=0}^{\infty} p_n(t) \varphi_n(x) \tag{26}$$

The time factors again satisfy Eqs.(22), (23) and, (24) with U_n replaced by φ_n, and the solution is given by Eq.(25) with U_n replaced by φ_n.

4. Results and Discussion

For both cases (1) and (2), five mode shapes were used in evaluating the response. The calculations are performed analytically using Wolfram Mathematica. The calculations for case (2) are shown in Figure 2 as an example. Table 1 lists the eigenvalues for the uncontrolled and the two controlled cases mentioned above for comparison.

```
K = 2;
x1 = 1 / 3;
x2 = 2 / 3;
w = ww /. NSolve[ww Sin[ww] ==
      -K Sin[ww/3] Sin[ww/3] && 0 < ww < 16, ww]
u = Sin[w[[k]] (x2 - x) ] HeavisideTheta[x - x2]
      + Sin[w[[k]] (1 - x2) ] / Sin[w[[k]]] Sin [w[[k]] x];
A = {};
For[k := 1, k < Dimensions[w][[1]] + 1, k++,
  Print[ "k=", k];
  vk = u - Sum[( ∫_0^1 ((u) * (vj)) dx / ∫_0^1 ((vj) * (vj)) dx ) * (vj), {j, 1, k-1}];
  bb = DSolve[{p''[t] + (w[[k]])^2 p[t] ==
        -Cos[Pi * t] * ∫_0^1 Sin[Pi * x] * vk dx / ∫_0^1 vk * vk dx,
      p[0] == 0, p'[0] == 0}, p[t], t];
  sol = p[t] /. bb;
  pk[t] = sol[[1]];
  AppendTo[A, {k, vk, pk[t]}];]
U = Sum[A[[n, 2]] * A[[n, 3]], {n, 1, Dimensions[w][[1]]}]
```

Figure 2. A Sample calculation.

Table 1. Eigenfrequencies for uncontrolled and controlled string

Uncontrolled	Case 1	Case 2
3.141593	2.67206	3.64467
6.283185	6.49607	6.00458
9.424778	9.42478	9.42478
12.566371	15.6081	12.4511
15.707963	18.8496	15.7996

Figure 3 shows the uncontrolled time-response for the mid-point of the string which blowsup while making sinusoidal vibrations.

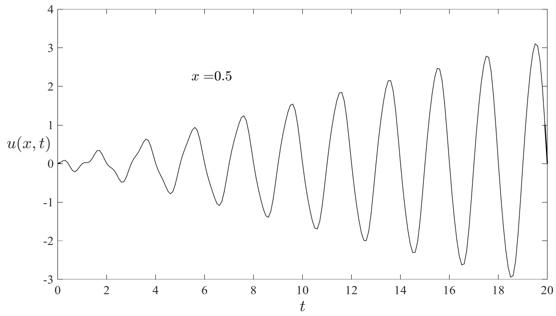

Figure 3. Uncontrolled response to sinusoidal forcing

Figure 4 shows the displacement in time of the mid-point of the string when the measurement and the actuation are at the same point $x_1 = x_2 = 1/3$. The unbounded growth has been suppressed and the mid-point makes periodical vibrations in time.

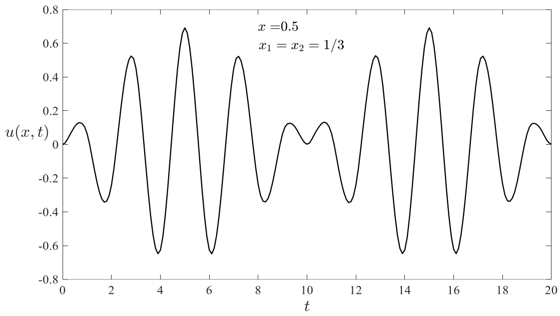

Figure 4. Controlled response when measurement and actuation are at the same point

Figure 5 and Figure 6 show the shape of the string at various times; there is a sharp change in slope at $x = x_1 = x_2 = 1/3$.

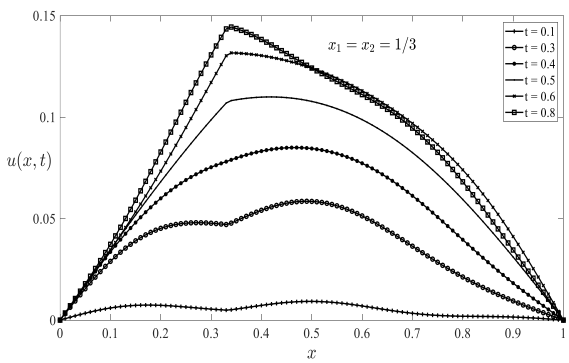

Figure 5. Shape of string at various times; controlled case (t $<$ 1)

Figure 7 shows the displacement in time of the mid-point of the string when the measurement is at $x_1 = 1/3$ and the actuation is at $x_2 = 2/3$. The response is similar to the case (1).

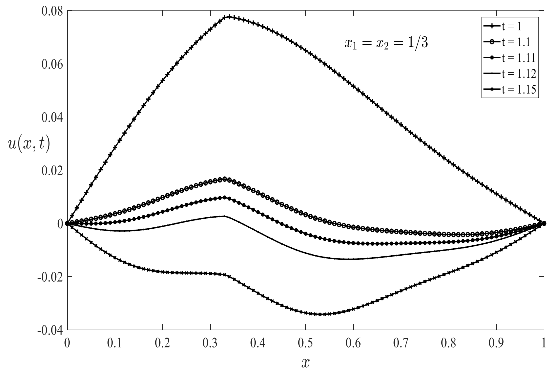

Figure 6. Shape of string at various times; controlled case (t $\geq$ 1)

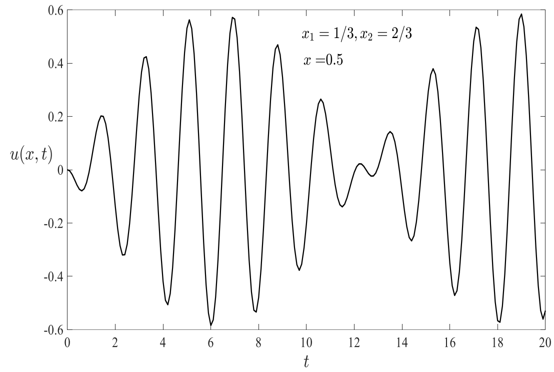

Figure 7. Controlled response when measurement and actuation are at different points

Similarly, Figure 8 and Figure 9 show the shape of the string at various times. In this case, there is a sharp change in slope at $x = x_2 = 2/3$ where the control is applied. But the string shape is smooth at $x = x_1 = 1/3$ where the measurement is made.

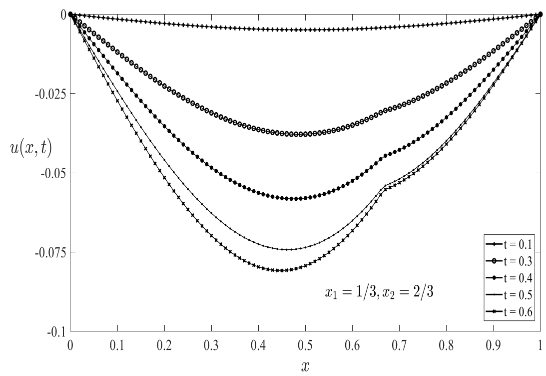

Figure 8. Shape of string at various times; controlled case (t $<$ 1)

Finally, a few remark concerning the control effort (the size of the controlling force) should be made. This quantity is $K\,u(x_1,t)$ and K was taken to be 2. Fig. 10 shows this for the case measurement and actuation points are the same, i.e., $K\,u\left(\frac{1}{3},t\right)$. Control force is basically twice the diplacement, but since the controlled displacement is smaller, the control effort also gets smaller.

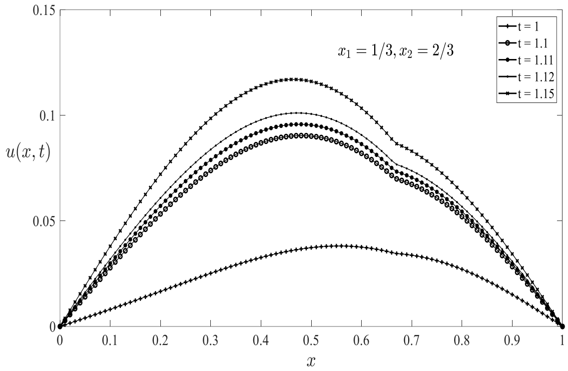

Figure 9. Shape of string at various times; controlled case ($t \geq 1$)

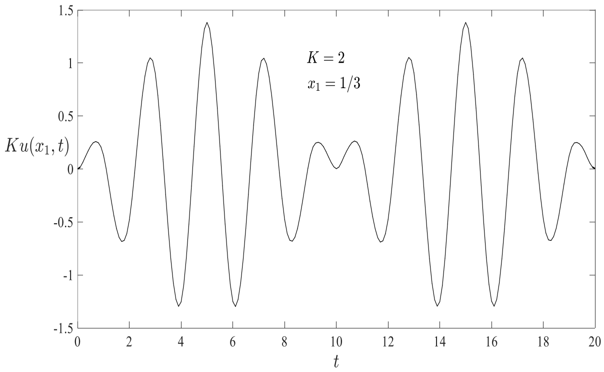

Figure 10. Control force as a function of time when actuation and measurement are at x=1/3

The controlled response can further be shaped by varying the constant K, the measurement and actuation points and more importantly, utilizing pointwise velocity-feedback, i.e., adding another control term to the right-hand side of Eq. (5) or (20) proportional to the time derivative of the displacement at possibly another point.

5. Conclusion

The vibrations of a string were controlled by applying the pointwise control concept: the string displacement was measured at a single point and a force proportional to this displacement is applied at another (or the same) point. The resulting controlled wave equation was solved analytically. The usual solution procedure namely, eigenfunction-expansion method modified and resulting non-orthogonal eigenfunctions were orthogonalized using the Gram-Schmit procedure. This analytical solution allows to carry out numerical experiments on the system for various input forms. As an example, the system was forced by an input that drives the uncontrolled system to resonance, causing growing sinusoidal vibrations. The controlled system suppresses the growth in vibrations and results in a bounded periodic output. As a result, it can be inferred that resonances can be avoided by the control procedure outlined here. Furthermore, it should be emphasized that pointwise control is a more realistic method to control such distributed parameter systems as considered here since it is impossible to actually measure the displacement field, as well as apply control forces, in a continuous interval. Only one measurement and one actuation point were considered here. By measurement and actuation at more points, better results may be obtained

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